Exercise 1.1
Mathematics 11 (PECTAA)

11-08-2025

Unit # 1: Complex Numbers

Complex Numbers: A number of the form $z = a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$, is called a complex number.
For example: $3 + 4i,\ 2 + i,\ -7 - 2i \in \mathbb{C}$ (set of complex numbers).

Notation: $z = a + bi$, where a is the real part and b is the imaginary part.

Examples: $2i,\ -3i,\ \sqrt{5}i,\ -4i^2$ are imaginary numbers.
For $3 + 4i$, real part = 3, imaginary part = 4.

Conjugate of Complex Numbers: Let $z = a + bi$ be a complex number. Then $a - bi$ is its complex conjugate, denoted by $\overline{z}$.

Example: $z_1 = 4 - 5i,\ \overline{z_1} = 4 + 5i$.

Operations on Complex Numbers

- **Addition:**
$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
- **Scalar Multiplication:**
$$k(a + bi) = ka + kbi$$
- **Subtraction:**
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$
- **Multiplication:**
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Properties of Complex Numbers

	Associative	**Additive Identity**
Addition	$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$	$z + 0 = z$
Multiplication	$z_1(z_2z_3) = (z_1z_2)z_3$	$z \cdot 1 = z$
	Additive Inverse	**Multiplicative Inverse**
	$\forall z = a + bi,\ \exists -z = -a - bi$	$\forall z = a + bi \neq 0,\ \exists \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$

Argand Diagram

There is a one-to-one correspondence between complex numbers and points in the xy-plane.

Modulus of a Complex Number

The modulus of $z = a + bi$ is:

$$|z| = \sqrt{a^2 + b^2}$$

Exercise 2.1

1. Find the multiplicative inverse of the following complex numbers:

(i) $(-4, 7)$

$$\text{Multiplicative inverse} = \left(\frac{-4}{(-4)^2 + 7^2}, \frac{-7}{(-4)^2 + 7^2} \right) = \left(\frac{-4}{65}, \frac{-7}{65} \right)$$

Alternate method:

$$\frac{1}{-4 + 7i} \cdot \frac{-4 - 7i}{-4 - 7i} = \frac{-4 - 7i}{65} = \left(\frac{-4}{65}, \frac{-7}{65} \right)$$

(ii) $(\sqrt{2}, -\sqrt{5})$

$$\text{Multiplicative inverse} = \left(\frac{\sqrt{2}}{2 + 5}, \frac{\sqrt{5}}{2 + 5} \right) = \left(\frac{\sqrt{2}}{7}, \frac{\sqrt{5}}{7} \right)$$

(iii) Multiplicative Inverse of $(1, 0)$

For $z = (1, 0) = 1 + 0i$:

$$z^{-1} = \frac{1}{1 + 0i} = \frac{1}{1^2 + 0^2} - \frac{0}{1^2 + 0^2} i = 1 + 0i = (1, 0)$$

Thus, $(1, 0)^{-1} = (1, 0)$.

0.1 Question 2 (i)

Seprate into real and Imaginary part

$$\begin{aligned}
\frac{2 - 7i}{4 + 5i} &= \frac{(2 - 7i)(4 - 5i)}{(4 + 5i)(4 - 5i)} \\
&= \frac{2 \cdot 4 + 2 \cdot (-5i) + (-7i) \cdot 4 + (-7i)(-5i)}{4^2 - (5i)^2} \\
&= \frac{8 - 10i - 28i + 35i^2}{16 - (-25)} \\
&= \frac{8 - 38i - 35}{41} \\
&= \frac{-27 - 38i}{41} \\
&= -\frac{27}{41} - \frac{38}{41} i
\end{aligned}$$

(ii)

$$\begin{aligned}
\frac{(-2+3i)^2}{1+i} &= \frac{(-2+3i)(-2+3i)}{1+i} \\
&= \frac{(-2)^2+2(-2)(3i)+(3i)^2}{1+i} \\
&= \frac{4-12i+9i^2}{1+i} \\
&= \frac{4-12i-9}{1+i} \\
&= \frac{-5-12i}{1+i} \\
&= \frac{(-5-12i)(1-i)}{(1+i)(1-i)} \\
&= \frac{-5+5i-12i+12i^2}{1-i^2} \\
&= \frac{-5-7i-12}{2} \\
&= \frac{-17-7i}{2} \\
&= -\frac{17}{2}-\frac{7}{2}i
\end{aligned}$$

(iii)

$$\frac{i}{1+i} = \frac{i(1-i)}{(1+i)(1-i)} = \frac{1+i}{2} = \frac{1}{2}+\frac{1}{2}i$$

(iv) Try yourself
Question 3

$$\begin{aligned}
z = x + iy, &\quad \overline{z} = x - iy \\
\text{Assume } \overline{z} = z \implies & x - iy = x + iy \\
-iy &= iy \\
-iy - iy = 0 \implies & -2iy = 0 \implies y = 0 \\
\therefore z &= x \text{ is real} \\
\implies \overline{z} = z &\iff z \text{ is real}
\end{aligned}$$

Properties of Conjugate and Modulus

Question 4
For $z \in \mathbb{C}$, show that:
(i) $z + \overline{z} = 2\,\mathrm{Re}(z)$ Let $z = a + bi \Rightarrow \overline{z} = a - bi$

$$
\begin{aligned}
z &= x + iy, \quad \overline{z} = x - iy \\
z + \overline{z} &= (x + iy) + (x - iy) = 2x \\
\frac{z + \overline{z}}{2} &= \frac{2x}{2} = x \\
&= \mathrm{Re}(z)
\end{aligned}
$$

$$
\begin{aligned}
z &= x + iy, \quad \overline{z} = x - iy \\
z - \overline{z} &= (x + iy) - (x - iy) = 2iy \\
\frac{z - \overline{z}}{2i} &= \frac{2iy}{2i} = y \\
&= \mathrm{Im}(z)
\end{aligned}
$$

(iii) $|z|^2 = z \cdot \overline{z}$

$$
z \cdot \overline{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2
$$

Question 5

Let $Z_1 = 2 + i,\ Z_2 = 3 - 2i,\ Z_3 = 1 + 3i$

Find $\frac{\overline{Z_1} \cdot \overline{Z_3}}{Z_2}$ in rectangular form.

$$
\begin{gathered}
\overline{Z_1} = 2 - i, \quad \overline{Z_3} = 1 - 3i \\
\frac{(2 - i)(1 - 3i)}{3 - 2i} = \frac{2 - 7i + 3i^2}{3 - 2i} = \frac{-1 - 7i}{3 - 2i} \\
= \frac{(-1 - 7i)(3 + 2i)}{(3 - 2i)(3 + 2i)} = \frac{-3 - 2i - 21i - 14i^2}{9 + 4} = \frac{11 - 23i}{13}
\end{gathered}
$$

Modulus Evaluation

Let $Z_1 = 2 + 7i,\ Z_2 = -5 + 3i$

$$
\begin{aligned}
Z_1 &= 2 + 7i, \quad Z_2 = -5 + 3i \\
2Z_1 &= 2(2 + 7i) = 4 + 14i \\
4Z_2 &= 4(-5 + 3i) = -20 + 12i \\
2Z_1 - 4Z_2 &= (4 + 14i) - (-20 + 12i) = 24 + 2i \\
|2Z_1 - 4Z_2| &= \sqrt{24^2 + 2^2} = \sqrt{576 + 4} = \sqrt{580} \\
&= 2\sqrt{145}
\end{aligned}
$$

(ii) $|3Z_1 + 2\overline{Z_1}| = |6 + 21i + 4 - 14i| = |10 + 7i| = \sqrt{149}$

$$
\begin{aligned}
(iii) |-7Z_2 + 2\overline{Z_2}| Z_2 &= -5+3i, \quad \overline{Z_2} = -5-3i \\
-7Z_2 &= -7(-5+3i) = 35-21i \\
2\overline{Z_2} &= 2(-5-3i) = -10-6i \\
-7Z_2 + 2\overline{Z_2} &= (35-21i) + (-10-6i) = 25-27i \\
|-7Z_2 + 2\overline{Z_2}| &= \sqrt{25^2 + (-27)^2} = \sqrt{625+729} = \sqrt{1354}
\end{aligned}
$$

(iv) $|(Z_1 + Z_2)^3| = |-3+10i|^3 = (\sqrt{109})^3 = 109\sqrt{109}$

Proof

Show that:

$$
i^{n+1} + i^{n+2} + i^{n+3} + i^{n+4} = 0 \quad \forall n \in \mathbb{N}
$$

$$
\begin{aligned}
\text{LHS} = i^{n+1} + i^{n+2} + i^{n+3} + i^{n+4} &= i^n(i + i^2 + i^3 + i^4) \\
&= i^n(i - 1 - i + 1) = i^n \cdot 0 = 0
\end{aligned}
$$