linear operator

def

let X and Y be two vector space over the same field K.
A linear operator $T: D(T) \to Y$ where $D(T)$ is a subspace of X is an operator $\ni$ $x, y \in D(T)$ and scalar α

(i) $T(x+y) = Tx + Ty$
(ii) $T(\alpha x) = \alpha(Tx)$

Clearly $T(\alpha x + \beta y) = \alpha Tx + \beta Ty$
If $D(T)$ is the whole space of X then we write $T: X \to Y$.
by taking $\alpha = 0$ in (ii) we obtain formula $T0 = (0)$ $\because \alpha = 0$
$\Rightarrow$ $T: X \to Y$ if $T(0) \neq 0$ then T is not linear

Identity operator

$$I(\alpha x + \beta y) = \alpha x + \beta y = \alpha I(x) + \beta (I)(x)$$

$x, y \in X$ and scalar α, β

Zero operator

The zero operator $0: X \to Y$ is linear

$$0x = 0 \quad \forall x \in X$$

$$0(\alpha x + \beta y) = 0 = \alpha 0(x) + \beta 0(y)$$

Differentiation

let X be the vector space of all polynomials in $[a,b]$ we may define a linear operator T on X by

$$Tx(t) = x'(t) \quad \forall x \in X$$

$$T(ax(t)+By(t)) = (ax(t)+By(t))'$$

$$= ax'(t) + By'(t)$$

$$= aTx(t)\ BTy(t)$$

Integration

A linear operator $T : C[a,b] \to C[a,b]$

$Tx = y$ where $y(t) = \int_a^t x(t)dt$

$t \in [a,b]$

α, B are Scalar

$$\int_a^t ax_1(t) + Bx_2(t)dt \leq a\int_a^t x_1(t)dt + B\int_a^t x_2(t)dt$$

Xing byt

$Tx = tx$ linear operator

$C[a,b]$

Elementary vector algebra

The cross product with one vector $\alpha = (\alpha_1, \alpha_2) \in R^2$

$T_1 : R^2 \to R^2$ define $T_1x = \alpha \times x$

$T_2 : R^3 \to R$ say $T_2x = x \cdot a = x_1\alpha_1 + x_2\alpha_2$

Metrices

A real matrices $A = \overline{a_{ij}}$ with m rows and n-columns

$$T : R^n \to R^m$$

$$Tx = Ax$$

T is linear because matrix xing is a linear operation

Null space

Let $T : D(T) \to Y$ be linear operator Null space $N(T)$

$$N(T) = \{ x \in D(T) = T(x) = 0 \}$$

Range and null space

(1) The Range $R(T)$ is a vector space.

(b) If $\dim D(T) = n < \infty$, then $\dim R(T) < n$

Consequently, linear operator preserve linear dependence.

(c) The null space $N(T)$ is a vector space.

Injective or one to-one

A mapping $T : D(T) \to Y$ is one-to-one if different points in the domain have different images if for any

$$x_1, x_2 \in D(T), \quad x_1 \neq x_2 \Rightarrow Tx_1 \neq Tx_2$$

$$Tx_1 = Tx_2 \Rightarrow x_1 = x_2$$

In this case there exist the

$\dim R(T) =$

Range Domain

mapping $T^{-1} : R(T) \to D(T)$

$y_0 \in R(T)$ onto $x_0 \in D(T)$

for which $Tx_0 = y_0$ T^{-1} is called the inverse of T

$T^{-1}(Tx) = x$ $x \in D(T)$ and

$T(T^{-1})y = y$ for all $y \in R(T)$

The inverse of a linear operator exists iff the null space of the operator consist of the zero vector only (mapping - injective)

The

let X and Y be two vector space the same field K

let $T : D(T) \to Y$ be linear operator with Domain $D(T) \subset X$ and $R(T) \subset Y$

(i) $T^{-1} : R(T) \to D(T)$ iff $Tx = 0 \Rightarrow x = 0$

(ii) If T^{-1} exist that is linear operator

(iii) If $\dim D(T) = n < \infty$ and T^{-1} exist then $\dim R(T) = \dim D(T)$

Inverse of composition

let $T : X \to Y$ and $S : Y \to Z$ be bijective where X, Y, Z be vector space The inverse $(ST)^{-1} : Z \to X$ the composit ST exist

$(ST)^{-1} = T^{-1}S^{-1}$

v.v.v → **rank - nulity Theorem**

$$\dim D(T) = \dim R(T) + \dim N(T)$$
$$= \dim R(T) + 0$$
$$= \dim R(T)$$

Bounded linear operator

let X and Y be normed space $T : D(T) \to Y$ where $D(T) \subset X$ a linear operator.

$x, y \in X$ and scalar α

(i) $T(x+y) = T(x) + T(y)$
(ii) $T(\alpha x) = \alpha(Tx)$

linear

T is said to bounded if there is a positive real no c

$\|Tx\| \leq c\|x\|$ then it is bounded

let $x \neq 0$ $\|Tx\| \leq c\|x\|$ can be written as

$$\frac{\|Tx\|}{\|x\|} \leq c \quad \text{where } c$$

is bounded above, in R therefor its supremum exist and suprimum is $\|T\|$

$\left\{\frac{\|Tx\|}{\|x\|} \; x \in D(T) - \{0\}\right\}$

$$\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in D(T) - \{0\}\right\}$$

$$\sup \frac{\|Tx\|}{\|x\|} \le \|T\| \quad (x \ne 0)$$

$$\Rightarrow \|Tx\| \le \|T\|\|x\| \Rightarrow \boxed{\text{bounded linear operator}}$$

$\|T\|$ is the smallest possible value c by supremum definition.

The $\Rightarrow$ A bounded linear operator maps bounded sets in $D(T)$ onto bounded sets in Y

Exp

Identity operator

The identity operator $I: X \to X$ on a normed space $X \ne \{0\}$ is bounded and has norm $\|I\| = 1$

$$\|Ix\| = \|x\| \le c\|x\| \quad \text{for any } c \ge 1$$

$$\|I\| = \sup\left\{\frac{\|Ix\|}{\|x\|} : x \in D(T) - \{0\}\right\}$$

$$= \sup \frac{\|x\|}{\|x\|} : x \in D(T) - \{0\}$$

$$= \sup\{(1, 1, 1, \dots)\} = 1$$

Zero operator

The zero operator $0: X \to Y$ on a normed space X is bounded and $\|0\| = 0$

$\|0x\| = 0 \leq c\|x\|$ $(c > 0$

and

$$\|0\| = \sup\left\{\frac{\|0x\|}{\|x\|} = 0 : x \in D(T) - \{0\}\right\}$$

$= 0$

Differentiation operator

let X be the normed space of all polynomials on $[0,1]$ with norm $\|x\| = \max_{t \in [0,1]} |x(t)|$ a differentiation operator T is defined on X

$Tx = y$ where $y(t) = x'(t)$

This operator is linear but not bounded

Integral operator

an Integral $T : C[0,1] \to C[0,1]$

$Tx = y$ where $y(t) = \int_0^t x(T)$

This operator is linear and bounded.

The Finite dim

If a normed space X is finite dim space then every linear operator on X is bounded

=> Identity operator is continuous but unbounded

=> Piecewise function is bounded but discontinuous.

The continuity and boundedness

let $T : D(T) \to Y$ be linear operator where $D(T) \subset X$ and X, Y are normed space then

(a) T is continuous iff T is bounded

(b) If T is continuous at a single point it is continuous

The

=> let T be a bounded linear operator then

(a) $x_n \to x \implies Tx_n \to Tx$

(b) The null space $N(T)$ is closed.

The let $T : D(T) \to Y$ be a linear operator where $D(T)$ lies in a normed space X and Y is a Banach. The T has an extension $\tilde{T} : \overline{D(T)} \to Y$ where $\tilde{T}$ is a bounded linear operator of Norm $\|\tilde{T}\| = \|T\|$

=> The inverse of differential operator does not exist.