compactness and finite Dim

compact

Def X – compact

at least one convergent subseq is compact

$x_n$ has a convergent subseq is called compact

compact ∃ subseq + convergent ∃ convergent

e.g. R = not compact

$\{n\} = \{1, 2, 3 \ldots\}$ increasing + unbounded

VVV

⟹ A compact subset M of a metric is closed bounded

compact ⟹ closed + bounded

e.g. let M is compact

(a) M is closed

$\bar{M} = M$ $M \subseteq \bar{M}$

$\bar{M}$ (smallest superset)

$\bar{M} \subseteq M$

let $x \in \bar{M}$ ∃ $x_n$ in M

$x_n \to x$

by compact $x_{n_k} \to y \in M$

( y is subseq in M )

$x_{n_k} \to y$ ⟹ $x_n \to y$ — ①

$x = y \in M$ ⟹ $x \in M$

$\bar{M} \subset M$ ⟹ $M \subset \bar{M}$

$\bar{M} = M$ (M is closed)

Scanned with CamScanner

$\Rightarrow$ اگر Set Compact ہے تو وہ لازمی closed or bounded ہوگا

## Converse

V.V.V Imp

Closed + bounded = compact may or may not

e.g $X = l^2$

$R^n = \{x_1, x_2 \dots x_n\}$

$\|\cdot\|_2 = \left(\sum_{i=1}^{n} \|x_i\|^2\right)^{1/2}$

$l^2 = \{x_1, x_2 \dots\}$

$\left(\sum_{i=1}^{\infty} |x_i|^2 < \infty\right)$

$\|x\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^2$ convergent

e.g $X$ is infinite

$\|e_n\| = (e_1^2 + e_2^2 \dots e_n^2 \dots)^{1/2}$

$(r)^{1/2} = 1$

$\|e\| \leq 1$ is bounded

$M$ is closed

$\bar{M} = M \cup M^d$

let $x_n \in M$ $\quad x_n \to x$

$\{e_1, e_2 \dots\}$

$e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$

$e_3 = (0, 0, 1) \dots$ $\quad M^d = \varphi$

$\bar{M} = M \cup M^d \Rightarrow \bar{M} = M$

then $M$ is closed and bounded

Now we check $M$ is compact

let $\{e_1, e_2 \dots\}$ has no convergent subsequence $\Rightarrow M$ is not compact

Scanned with CamScanner

R) $X$ is finit dim. iff closed and bounded (compact)

e.g. $R' \Rightarrow [a;b]$ closed + bounded

($l.b = a$ , $u b = b$) = compact

F. Riezz

$X$ – Norm Space

$Z < X$ $Y$ is closed Subspac

$Y \subseteq Z$ $\forall \theta$; $\exists$ $0 < \theta < 1$

$z \in Z$ $\|z\| = 1 \in R$

$\|z - y\| \geq \theta$ for all $y \in Y$

The 9.D. V.V.V. Dim D

If a Normed Space $X$ has the property that the closed unit ball $M = (x \mid \|x\| < 1)$ is compact then $X$ is finit dim

proof Let $M$ is compact

but $X$ = Infinit dim

$\dim X = \infty$

let $\|x_1\| = 1$ $X_1 = \text{Span} = \{x\} \Rightarrow$ vector = closed + finite

$X_1 \in X_\infty$

$\|z - y\| \geq \frac{1}{2}$

$\|x_2 - x_1\| \geq \frac{1}{2}$ $z = x_2$

let $X_2 = \text{Span}\{x_1, x_2\}$

$\dim X_2 = 2$ (Finite) $\Rightarrow$ closed

$X_2 \subset X_\infty$ , $\theta = \frac{1}{2}$ $z = x_3$

$$||z - y|| \geq \frac{1}{2}$$

$$||x_3 - y|| \geq \frac{1}{2}$$

$$y \in sp\{x_1, x_2\}$$

$$||x_3 - x_1|| \geq \frac{1}{2}$$

$$||x_n - x_{n-1}|| \geq \frac{1}{2}$$

$\Rightarrow \{x_n\}$ in $M$ not cauchy.

**The**

Let $X$ and $Y$ be metric space and $T : X \to Y$ a continuous mapping. then the image of a compact subset $M$ of $X$ under $T$ is compact